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COMPLEX ZEROS OF A TRANSCENDENTAL IMPEDANCE FUNCTION

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SUMMARY

The electrical impedance of a semiconductor supporting two waves contains an entire transcendental function of the form $f(z)=\exp(-z)-1-cz$, where c is a complex parameter. This function has an infinity of zeros in the left half z-plane when c is finite ($0<|c|<+\infty$). Several approximate expressions for the location of zeros as function of c are obtained. For certain values of c (cf. Fig. 3) one or several zeros are located in the right half z-plane. The real part of some of those exceeds an arbitrarily large positive number, provided c is properly chosen. This corresponds to resonances which represent growing oscillations.

1. *Introduction*

A uniform semiconductor subjected to a strong d.c. electric field can support various types of high frequency waves. Assume we have two waves of different types with small amplitudes and equal frequencies, and suppose that one of the waves is of electromagnetic type with vanishing wave number (in the quasistatic approximation). Further, assume that the other wave does not carry any total current and that its wave number is proportional to the frequency, $k=\omega/v$. (Examples of such a wave are a phonon wave or a carrier wave). Then, a slice of this material has an electric impedance (in dimensionless units), [1, 2].

$$Z = \frac{\exp(-z)-1-cz}{-cz^2} \tag{1.1}$$

Here $z=ik\ell$, ℓ is the thickness of the slice, and c is a complex constant which represents the amplitude ratio and phase difference between the two waves. If the device is short-circuited, the zeros of the impedance function determine the resonant frequencies. The location of a zero in the complex frequency plane determines whether the corresponding resonance represents a growing oscillation or not.

We are thus faced with the problem of discussing the zeros of the function

$$f(z) = \exp(-z)-1-cz \tag{1.2}$$

where c is a complex constant. For the case $c = -1$ this problem has been solved by McCumber and Chynoweth [3].

The only singular point of the function (1.2) is $z = \infty$, which is an isolated essential singularity. Thus, f(z) is an entire, transcendental function, which has the following power series,

$$f(z) = -(c+1)z + \frac{1}{2!}z^2 - \frac{1}{3!}z^3 + \dots \tag{1.3}$$

From this it appears that f(z) has the zero $z = z_0 = 0$. The order of the zero is two if $c = -1$. Otherwise the zero is of first order.

We shall see that f(z) has a denumerable number of zeros. Except for the zero at $z = 0$, the location of any zero varies with the complex parameter c.

Thus, the zeros are represented by a many-valued function

$$z_o = z_o(c), \tag{1.4}$$

which is the solution of the equation

$$f(z_o) = \exp(-z_o)-1-cz_o = 0 \tag{1.5}$$

or the inverse of the function

$$c = \frac{\exp(-z_o)-1}{z_o} \tag{1.6}$$

Our purpose here is to discuss the position of the zeros. In particular, we shall examine for which values of c zeros are located in the right half z-plane (Re $z_o > 0$).

Occasionally, we shall find it convenient to write

$$c = a+ib \quad \text{and} \quad z = x+iy \tag{1.7}$$

where a, b, x, and y are real numbers.

2. *Zeros in the real domain*

Setting z = x and c = a, the condition for real zeros becomes

$$\exp(-x_o)-1 = ax_o \tag{2.1}$$

If a > 0 the only real root of this equation is

$$x_o = 0 \tag{2.2}$$

If a < 0 there is an additional real root. It is negative if a < -1 and positive if -1 < a < 0 (cf. Fig. 1).

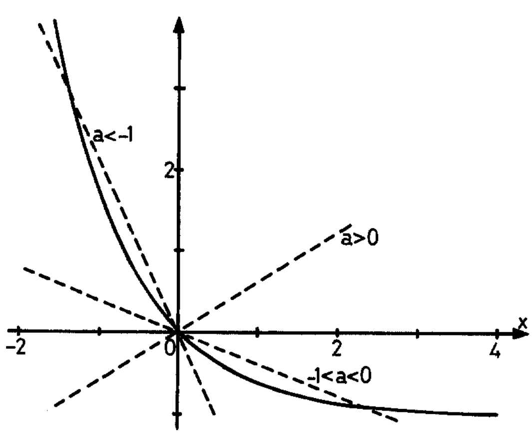

Fig. 1 Plots of exp(-x)-1 (fully drawn curve) and of ax (dashed line). The points of intersection determine the real zeros of f(z).

This demonstrates that a real zero passes along the real z-axis (the real axis of the z-plane) from $-\infty$ to $+\infty$ when the parameter c passes along the real c-axis from $-\infty$ to 0.

3. *Zeros when* $|c|$ *is small*

For the case $c = 0$, $f(z)$ evidently has the zeros

$$z_o = z_o(0) = i2\pi\nu, \qquad \nu = \ldots, -2, -1, 0, 1, 2, \ldots \tag{3.1}$$

Differentiating (1.6), we get

$$dc = \frac{-(z_o+1)\exp(-z_o)+1)}{z_o^2}\, dz_o \tag{3.2}$$

Inserting (3.1) this gives

$$\frac{dz_o(0)}{dc} = -z_o = -i2\pi\nu \tag{3.3}$$

Hence, for $c \to 0$ we have

$$z_o = z_o(c) = i2\pi\nu - i2\pi\nu c + O\{c^2\} \tag{3.4}$$

For every ν this represents a power series in c. Since $z_o(c)$ is a many-valued function, branch points exist in the c-plane, and for $\nu \neq 0$ every power series (3.4) has a finite radius of convergence. (Cf. Section 8 - Appendix). Hence, on the circle of convergence there is a singular point $c = c_\nu$ at which dz_o/dc does not exist. From (1.5) and (3.2) we find

$$\frac{dz_o(0)}{dc} = -\frac{z_o}{cz_o+c+1} \tag{3.5}$$

If we let $c \to c_\nu$ such that $|c|<|c_\nu|$, we can use (3.4). Then, for $z_o \neq \infty$ we find

$$\frac{dz_o(c)}{dc} \to \infty \quad \text{when} \quad 1+(1+z_o)c = 1+(1+i2\pi\nu)c - i2\pi\nu c^2 + O\{c^3\} \to 0 \tag{3.6}$$

This condition determines the singular (branch) point c_ν. If $|c_\nu|$ is sufficiently small, that is, when $|\nu|$ is sufficiently large, we see from (3.6) that

$$c_\nu \approx -1/(1+i2\pi\nu + \frac{i2\pi\nu}{1+i2\pi\nu}) \approx \frac{i}{2\pi\nu} \tag{3.7}$$

This agrees with (8.11). Cf. also (8.8).

We can conclude that a zero is given approximately by the first two terms of the right hand side of (3.4), provided

$$|c| \ll \frac{1}{2\pi|\nu|} \tag{3.8}$$

Then we have

$$\mathrm{Re}\; z_o \approx 2\pi\nu\; \mathrm{Im}\; c = 2\pi\nu b, \tag{3.9}$$

which means that if $|c| \ll 1$ and Im $c \neq 0$ there is at least a finite number of zeros in the right half z-plane.

For completeness, we here mention one particular zero for $|c| \ll 1$ not included in the preceding discussion of the present section. For this case, refer to (5.14).

4. *Zeros near the origin of the z-plane*

Apart from the fixed zero $z_0 = 0$, there may be another zero near the origin of the z-plane, provided $|c+1|$ is sufficiently small. The first three terms of the series (1.3) give for this zero

$$z_0 = 2(1+c) + \frac{4}{3}(1+c)^2 + O\{(1+c)^3\} \tag{4.1}$$

For $|1+c| \ll 1$ we thus have

$$\mathrm{Re}\ z_0 \approx 2(1+a) + \frac{3}{4}(1+a)^2 - \frac{4}{3}b^2 \tag{4.2}$$

This zero is in the right half z-plane if

$$(1+a)^2 + \frac{3}{2}(1+a) > b^2 \tag{4.3}$$

In Fig. 3 the edge of this hyperbolic region is indicated by the dashed curve.

5. *Zeros far from the origin of the z-plane*

Eq. (1.5) can be rewritten as

$$z_0 = -\ln(1+cz_0) = \tag{5.1}$$

$$= -\ln|1+cz_0| - i\arg(1+cz_0) = \tag{5.2}$$

$$= i2\pi\nu - i\,\mathrm{Arg}(1+cz_0) - \ln|1+cz_0| \tag{5.3}$$

where

$$\nu = \ldots, -2, -1, 0, 1, 2, \ldots \tag{5.4}$$

and

$$-\pi < \mathrm{Arg}(1+cz_0) \leq \pi \tag{5.5}$$

We see that if $c = 0$, this result agrees with (3.1).
If

$$|cz_0| \gg 1, \tag{5.6}$$

iterative calculation for the solution of (5.3) converges rapidly. As a first approximation we take

$$z_0 \approx i2\pi\nu \tag{5.7}$$

which in view of (5.6) is a fairly good approximation if

$$|\nu| \gg \frac{1}{2\pi|c|} \quad \text{or} \quad |c| \gg \frac{1}{2\pi|\nu|} \tag{5.8}$$

As a better approximation we then have from (5.3)

$$z_0 \approx i2\pi\nu - i\,\mathrm{Arg}\,(1+i2\pi\nu c) - \ln|1+i2\pi\nu c| \tag{5.9}$$

$$\approx i2\pi\nu - i\frac{\pi}{2}\frac{|\nu|}{\nu} - i\,\mathrm{Arg}\ c - \ln|2\pi\nu c| \tag{5.10}$$

this expression gives the approximate location of infinitely many zeros. None of them lies in the right half part of the z-plane, for

$$\mathrm{Re}\ z_0 \approx -\ln|2\pi\nu c| < 0 \tag{5.11}$$

because of (5.8).

The relative error of the approximation decreases as $|\nu|$ increases. Better approximation can be obtained, if necessary, by further iteration on (5.3). For $\nu = 1$ and $c = -1$, the approximation (5.9) gives $z_0 \approx -1.85+i7.70$, while the correct numerical value is $z_0 = -2.09+i7.46$, [3].

We observe that for $\nu = 0$, the condition (5.8) can not be fulfilled for any finite c. If $|cz_0| \gg 1$ we can still apply (5.3) for an iteration procedure, but instead of using (5.7) we start the iteration procedure for $\nu = 0$ by a z_0 which is a negative number of sufficient magnitude. The solution corresponds to a generalization of the zero discussed in Section 2 when $a = \mathrm{Re}\ c$ has a large negative value. The iteration procedure will not be carried out here, because it concerns a zero in the left half z-plane ($\mathrm{Re}\ z_0 < 0$).

Eq. (5.3) shows that if $|c|$ is sufficiently small, there exists a zero with $\mathrm{Re}\ z_0 \gg 1$ if $|1+cz| \ll 1$. This requires that $\mathrm{Re}\ c < 0$.
In this case (5.3) is not well suited for iteration. Instead, we rewrite (1.5) as

$$z_0 = \frac{1-\exp(-z_0)}{-c} \tag{5.12}$$

A first approximation

$$z_0 \approx -\frac{1}{c} \tag{5.13}$$

and, iterating gives the better approximation

$$z_0 \approx \frac{1-\exp(1/c)}{-c} \tag{5.14}$$

valid when

$$\frac{-\mathrm{Re}\ c}{|c|^2} = \mathrm{Re}\ \left(\frac{1}{c}\right) \gg 1 \tag{5.15}$$

This zero corresponds to a generalization of the case discussed in Section 2 when a is a small negative number. An interesting point is that by appropriate choice of c this zero can have an arbitrarily large real part.

6. *Zeros having positive real part*

We have seen that it is possible to have a zero in the right half z-plane if $-1 < \mathrm{Re}\ c < 0$ and $\mathrm{Im}|c|$ is not too large. - Cf. Section 2 and Equation (5.14). - Further, (3.8) and (3.9) show that there may be any (natural) number of zeros in the right half z-plane, provided $|c|$ is sufficiently small and Im $c \neq 0$. For a given c, however, the number of zeros in the right half z-plane is finite - cf. (5.8) and (5.9).

In order to find conditions on c for satisfying Re $z_0 > 0$, we can apply "the principle of the argument" [4]:

$$N = \frac{1}{2\pi} \Delta_C \arg f(z) \tag{6.1}$$

In words: Provided a function f(z) has no singular point on and inside a closed contour C, the number N of zeros (counted m times if the order of the zero is m) inside this contour equals $(1/2\pi)$ times the change in argument of f(z) as z moves once around the closed contour. Of course, N is an integer, when no zero lies on the contour. Application of (6.1) to the contour C of Fig. 2 when X = 0 and $R \to +\infty$ yields the famous Nyquist stability criterion.

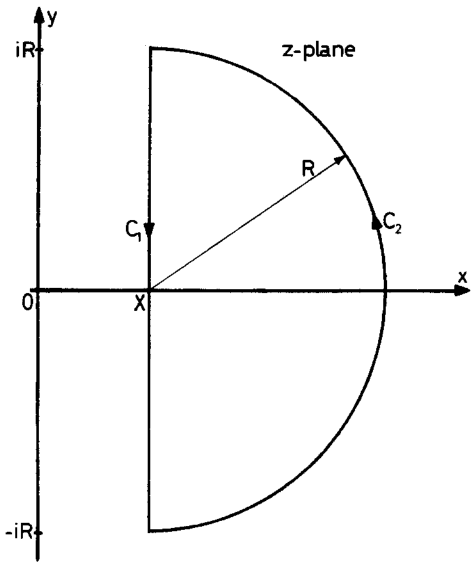

Fig. 2 Closed contour $C = C_1 \cup C_2$ in the complex z-plane, where $C_1 = \{z = X + iy \mid R > y > -R\}$ and $C_2 = \{z = X + R\exp(i\phi) \mid -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\}$. In the considered cases $X \geq 0$, and $R \to +\infty$.

Since our function f(z) always has a zero at z = 0 we exclude this zero by defining a new function

$$g(z) = \frac{f(z)}{z} = \frac{\exp(-z)-1}{z} - c \tag{6.2}$$

On the contour C_2 of Fig. 2 (the semicircle),

$$|g(z)+c| \leq \frac{2}{|z|}$$

since $|\exp(-\underline{z})| \leq 1$ there, provided $X \geq 0$. Therefore

$$g(z)\big|_{z\in C_2} \to -c \quad \text{as} \quad R \to +\infty \tag{6.3}$$

Hence, as z moves along C_2 $(R \to +\infty)$ there is no change in the argument of g(z),

$$\lim_{R \to +\infty} \Delta_2 \arg g(z) = 0 \tag{6.4}$$

Using (1.7), we have on the contour C_1 of Fig. 2, $x = X$,

$$\mathrm{Re}\; g = A(y) - a \tag{6.5}$$

$$\mathrm{Im}\; g = B(y) - b \tag{6.6}$$

where

$$A(y) \equiv \frac{-X\{1-\exp(-X)\cos y\} - \exp(-X) y \sin y}{X^2 + y^2} \tag{6.7}$$

$$B(y) \equiv \frac{\{1-\exp(-X)\cos y\} y - X\exp(-X)\sin y}{X^2+y^2} \tag{6.8}$$

Setting $g = 0$ on C_1 (i.e. X is fixed) yields

$$a = A(y) \quad \text{and} \quad b = B(y) \tag{6.9}$$

which are the equations for a plane curve in parametric form (y is the parameter). Such curves are exhibited in Figs. 3, 4, and 5 for $X = 0$, $X = 1$, and $X \gg 1$, respectively.

Apparently, $A(y)$ is an even function of y, and $B(y)$ is an odd function. For the case $X = 0$, (6.7) simplifies to

$$A(y) = -\frac{\sin y}{y} \tag{6.10}$$

which has the minimum value

$$A_o = -1 \quad \text{at} \quad y = 0 \tag{6.11}$$

and maximum value

$$A_1 \approx 0.2172 \quad \text{at} \quad y \approx \pm 4.49 \tag{6.12}$$

Further, for $X = 0$, (6.8) simplies to

$$B(y) = \frac{1-\cos y}{y} \tag{6.13}$$

which has the maximum (minimum) value

$$\underset{(-)}{+} B_0 \approx \underset{(-)}{+} 0.7246 \quad \text{at} \quad y \approx \underset{(-)}{+} 2.33 \tag{6.14}$$

Consequently, if $a < -1$ or $a > A_1$, Re g never changes sign on C_1. Hence, $g(z)$ has no net change in argument as z describes the contour $C_1 (R \to +\infty)$, that is $\Delta_1 \arg g = 0$. Or, if $|b| > B_o$, Im g never changes sign on C_1, which means that $\Delta_1 \arg g = 0$ also in this case. With (6.1) and (6.4) in mind, we see that $g(z)$ and, hence, $f(z)$ have no zero in the right half z-plane if $a < -1$, or $a > A_1 \approx 0.2172$, or $|b| > B_o \approx 0.7246$.

If on the other hana

$$-1 < a < A_1 \quad \text{and} \quad -B_o < b < B_o, \tag{6.15}$$

then $f(z)$ may, or may not, have zeros in the right half z-plane depending

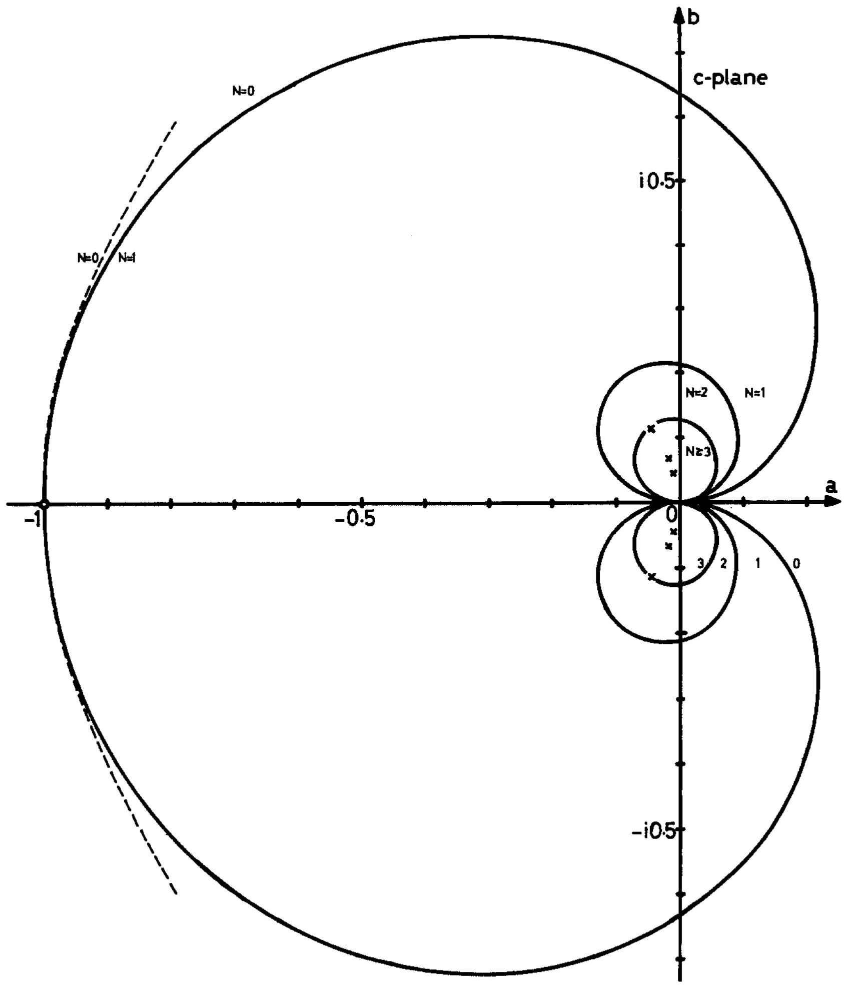

Fig. 3 The number N of zeros of f(z) in the right half z-plane depends on the complex parameter c as indicated in this diagram, for N=0, N=1, N=2 and N $\geq$ 3. The six points marked by crosses (x) are branch points for the function $z_0(c)$. They are, c_1, c_2, c_3, c_{-3}, c_{-2}, and c_{-1} (listed from above and downwards in the figure). The points $c_0 = -1$ (not a branch point) is marked with a small circle (o). Cf. (3.7) and Table 1. The hyperbola defined by (4.3) is indicated by the dashed curve.

on how a and b are interrelated, as shown in Fig. 3.

The curve given by (6.9) when X = 0 are composed of an infinity of closed (sub-)curves which touch each other at c = 0 as indicated in Fig. 3. These curves divide the c-plane into regions in such a way that if c belongs to a particular region, then f(z) has a certain number N of zeros in the right half z-plane. Fig. 3 displays the curves which separate regions giving N = 0, 1, 2, 3. For N $\geq$ 2 the regions are shaped like a figure of eight, with decreasing extension as N increases. The curves cross the imaginary axis of the c-plane in the points $c = i2/\{(2k+1)\pi$, $k = \pm1, \pm2, \pm3, \ldots$ corresponding to $y = (2k+1)\pi$. In accordance with (3.9) we find that f(z) may have any (natural) number of zeros in the right half z-plane, provided $|c|$ is small enough and Im c $\neq$ 0.

The curves in Fig. 3 represent a mapping of a straight line between the points $z = 6\pi i$ and $z = -6\pi i$ in the z-plane. Each separating curve, as shown in Fig. 3, represents a mapping of a part of the imaginary axis of the z-plane, by an appropriate branch of the function $z_0(c)$. Suppose c moves along the curves in the clockwise direction starting at $c = -1$. Then a zero moves from $z = 0$ upwards along the axis $\mathrm{Re}\, z = 0$. When c is at $\mathrm{Re}\, c = 0$ the zero is at $z = \pi i$. Further when c approaches $c = 0$ the zero approaches $z = 2\pi i$. (Cf. (3.4) for $\nu = 1$.) Continuing on the curve which separates the "N= 1" and "N=2" regions of Fig. 3 until c again approaches $c = 0$, the zero approaches $z = 4\pi i$. (Cf. (3.4) for $\nu = 2$.) Letting c circle once more in the clockwise direction, now on the curve which separates the "N=2" and "N=3" regions, the zero moves from $z = 4\pi i$ to $z = 6\pi i$.

By the procedure just described, the many-valued-ness of the function $z_0(c)$ clearly appears. By each round trip of c (as described above) we find a new value of z_0 at $c = 0$, which means that a branch point of $z_0(c)$ has been encircled. This is discussed in more detail in Section 8 (Appendix). A few branch points are depicted in Fig. 3.

Also if $X > 0$, it can be shown that f(z) may have any number of zeros satisfying $\mathrm{Re}\, z_0 > X$ provided c is properly chosen. It is necessary that $|c|$ is very small. In fact, (5.3) shows that if, for any ν, c has such a value as to make $|1+cz_0|$ sufficiently close to zero, than $\mathrm{Re}\, z_0$ may be arbitrarily large.

The curve given by (6.9) for $X = 1$ makes one loop around each branch point c_μ, where $\mu = \pm 1, \pm 2, \pm 3, \ldots$. A few of the loops appear in Fig. 4. If c belongs to one of the regions bounded by the two loops around c_1 and c_{-1}, then f(z) has two and only two zeros satisfying $\mathrm{Re}\, z_0 > 1$. The regions bounded by the other loops partly overlap each other. There are n such zeros (N=n) if c is located where (n-1) "loop regions" overlap each other.

If $X \gg 1$, exp(-X) is an extremely small number. For $|y| \ll \exp(X)$ we then neglect exp(-X) in (6.7) and (6.8). We can then eliminate y from the two equations (6.9), and there remains

$$\left(a + \frac{1}{2X}\right)^2 + b^2 = \left(\frac{1}{2X}\right)^2 \qquad (6.16)$$

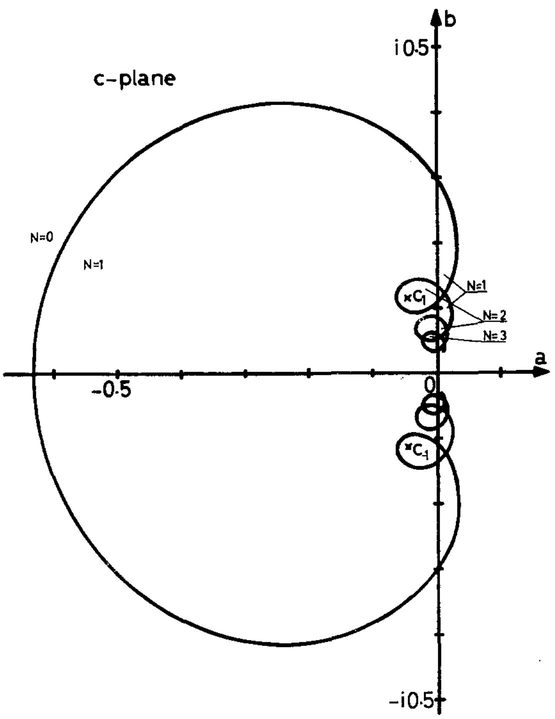

Fig.4 The number N of zeros of f(z) in the region Re z > 1, depends on the complex parameter c as indicated in this diagram.

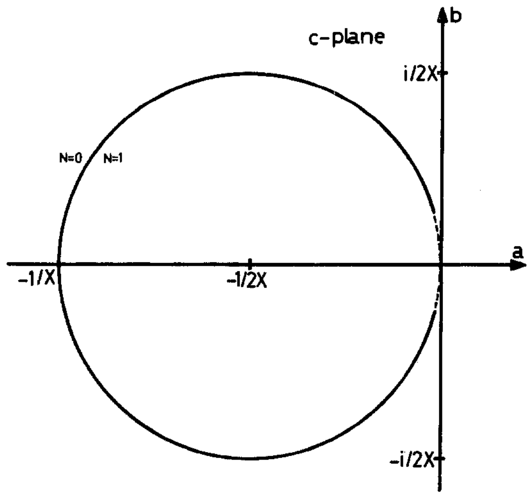

Fig. 5. When $c = a + ib$ is a point inside (outside) the shown circle, then f(z) has one (no) zero z_0 satisfying Re $z_0 > X$, provided $X \gg 1$ and $|c| \gg \exp(-X)$. Cf. (6.16).

When c is inside this circle (cf. Fig. 5), f(z) has one zero with Re $z_0 > X$. This zero is the one that is given approximately by (5.14). When $|y|$ is comparable with or larger than exp(X) the corresponding point on the curve (6.9) has $|c|$ comparable with or smaller than exp(-X). Then the approximation (6.16) no longer represents the details of the curve. In fact, the curve makes loops similar to those of the curve for the case X = 1, but only around the branch points $c_{\pm k}$, where $k = p, p+1, p+2, \ldots$ and p is the smallest natural number such that Re $z_{o,p} > X$ where $z_{o,p}$ is a double zero of f(z) when $c = c_p$ (cf. Section 8 - Appendix). Using (8.4) we find an estimate for the positive integer p,

$$p \approx \frac{\exp(X)}{2\pi} \tag{6.17}$$

If c belongs to a certain neighbourhood of one of the branch points $c_{\pm k}$, then at least two zeros of f(z) has Re $z_0 > X$. Cf. (8.10). The extent of the neighbourhood is bounded by a loop of the curve (6.9). When k is sufficiently large, an arbitrarily large number (n-1) of loops overlap each other, and the number of zeros with Re $z_0 > X$ is correspondingly large (= n).

7. *Conclusion*

We have shown that the function $f(z) = \exp(-z)-1-cz$ has infinitely many zeros. They are, however, denumerable. Except for a zero at z = 0, the location of the zeros varies with the complex parameter c.

Representations for the location of the zeros are given by several series expressions or approximations valid for different cases: for small $|c|$, for small $|z|$, and for large $|z|$.

Further it has been shown that if c is properly chosen (cf. Fig. 3), f(z) has one or more zeros in the right half z-plane. If $|c|$ is finite ($0 \ll |c| < +\infty$), the number of such zeros is finite. Thus, there are always infinitely many zeros in the left half z-plane.

It has been observed that f(z) may have one zero with arbitrarily large positive real part, provided Re $c < 0$ and $|c|$ is small enough (cf. Fig. 5). We choose a large positive number X ($X \gg 1$). We have found that f(z) has at least two zeros with Re $z_0 > X$ if c belongs to a certain neighbourhood of one of those branch points c_μ for which $|c_\mu|$ is less than a number approximately equal to $\exp(-X)$.

The above findings mean that by proper adjustment of amplitude ratio and phase difference between the two waves mentioned in the Introduction it is possible to obtain resonances which represent growing oscillations.

8. (APPENDIX) *Zeros of second order. Branch points for $z_0(c)$*

From (1.3) it appears that $z = 0$ is a zero of second order if $c = -1$. Otherwise the zero is of first order.

The function $f(z)$ has the following Taylor series around a zero z_0,

$$f(z) = -(c+\exp(-z_0))(z-z_0)+\exp(-z_0)\left\{\frac{(z-z_0)^2}{2!} - \frac{(z-z_0)^3}{3!} + \ldots\right\} \tag{8.1}$$

Since $\exp(-z_0) \neq 0$ for $z_0 \neq \infty$, no zero can be of higher order than two. For a zero z_0 to be of second order it is necessary that

$$c = -\exp(-z_0) \tag{8.2}$$

Inserting this into (1.5) gives

$$\exp(z_0) - 1 - z_0 = 0 \tag{8.3}$$

which also means that the denominator in (3.2) vanishes. If z_0 is one of the solutions of (8.3) and c has the specific value given by (8.2), then, and only then, $f(z)$ has a zero of second order.

We see that $z_0 = 0$ and $c = -1$ is a solution of (8.3) and (8.2) as was to be expected. All the other (infinitely many) solutions of (8.3) lie in the right half z-plane and they appear to exist in complex conjugate pairs. We observe that the problem of solving (8.3) is identical with the problem of finding the zeros of (1.2) if we let $c = -1$ and $z = -z_0$.

We then can apply (5.9) to yield the following approximate expression for the solution of (8.3)

$$z_0 = z_{0,\mu} \approx \ln|1+2\pi\mu i| + i2\pi\mu + i\,\mathrm{Arg}(1+2\pi\mu i), \quad \mu = \pm 1, \pm 2, \ldots \tag{8.4}$$

From (8.2) we then know that $f(z)$ has a zero of second order if

$$c = c_\mu = -\exp(-z_{0,\mu}), \quad \mu = \ldots, -2, -1, 0, 1, 2, \ldots \tag{8.5}$$

where $z_{0,\mu}$ is a solution of (8.3). For every μ,

$$|c_\mu| \leq 1 \text{ and } -1 \leq \mathrm{Re}\, c_\mu < 0. \tag{8.6}$$

We have

$$c_0 = -1 \tag{8.7}$$

and, generally,

$$c_\mu = \frac{-1}{1 + z_{0,\mu}} \approx \frac{-1}{1 + i2\pi\mu} \tag{8.8}$$

If necessary, better approximation can be obtained by further iteration. Table 1 shows the results of iterative computations.

Table 1.

Solutions $z_{o,\mu}$ of (8.3) and branch points c_μ for the many-valued function $z_o(c)$. (However, c_o is an ordinary point.) When $c = c_\mu$, f(z) has a double zero at $z = z_{o,\mu}$. Cf. (8.4) to (8.8).

μ	$z_{o,\mu}$	c_μ	$(-\{1+i2\pi\mu\}^{-1})$
0	0	-1	(-1)
$\pm$ 1	2.089 $\pm$ i 7.461	-0.0474 $\pm$ i 0.1144	(-0.025 $\pm$ i 0.155)
$\pm$ 2	2.664 $\pm$ i 13.879	-0.0178 $\pm$ i 0.0674	(-0.006 $\pm$ i 0.079)
$\pm$ 3	3.026 $\pm$ i 20.224	-0.0095 $\pm$ i 0.0476	(-0.003 $\pm$ i 0.053)
$\pm$ 4	3.292 $\pm$ i 26.543	-0.0059 $\pm$ i 0.0367	(-0.002 $\pm$ i 0.040)
$\pm$ 5	3.501 $\pm$ i 32.851	-0.0041 $\pm$ i 0.0299	(-0.001 $\pm$ i 0.032)
$\pm$ 6	3.675 $\pm$ i 39.151	-0.0030 $\pm$ i 0.0252	(-0.001 $\pm$ i 0.027)
$\pm$ 7	3.822 $\pm$ i 45.447	-0.0023 $\pm$ i 0.0218	(-0.001 $\pm$ i 0.023)
$\pm$ 8	3.951 $\pm$ i 51.741	-0.0018 $\pm$ i 0.0192	($\pm$ i 0.020)
$\pm$ 9	4.065 $\pm$ i 58.032	-0.0015 $\pm$ i 0.0171	($\pm$ i 0.018)
$\pm$ 10	4.167 $\pm$ i 64.322	-0.0012 $\pm$ i 0.0154	($\pm$ i 0.016)

For comparison, results based on the approximation (8.8) are given in parentheses. (The approximation (3.7) is slightly closer to the correct value than the approximation (8.8).) A few of the points c_μ are shown in Fig. 3.

We can expand f(z) in the following two-dimensional Taylor series in $(c-c_\mu)$ and $(z-z_{o,\mu})$,

$$f(z) = -z_{o,\mu}(c-c_\mu) - (c-c_\mu)(z-z_{o,\mu}) - c_\mu\left\{\frac{(z-z_{o,\mu})^2}{2!} - \frac{(z-z_{o,\mu})^3}{3!} + \dots\right\} \tag{8.9}$$

Setting f(z) = 0, this gives for the zeros as $c \to c_\mu$ $(\mu \neq 0)$,

$$z_o = z_{o,\mu} + \left(-2\frac{z_{o,\mu}}{c_\mu}\right)^{1/2}(c-c_\mu)^{1/2} + \frac{1}{c_\mu}\left(\frac{1}{3}z_{o,\mu} - 1\right)(c-c_\mu) + O\left\{(c-c_\mu)^{3/2}\right\} \tag{8.10}$$

Since $z_{o,o} = 0$, the analogue of (8.10) for $\mu = 0$ is (4.1). We observe that for $\mu \neq 0$, every point c_μ is a branch point of order one for the many-valued function $z_o(c)$. That is, one particular c_μ is a singular point for two specific branches of $z_o(c)$, while it is an ordinary point for all the (infinitely many) other branches of $z_o(c)$. (The reader who is not familiar with such a case may consult the classical book of Knopp [5].) The point $c_o = -1$ is an ordinary point for all branches of $z_o(c)$.

From (8.8) or (3.7) we see that

$$c_\mu \to \frac{i}{2\pi\mu} \to 0 \text{ as } |\mu| \to +\infty \tag{8.11}$$

This demonstrates that c = 0 is a limit point of branch points. Nevertheless, (3.4) shows that c = 0 is an ordinary point for a denumerable set of branches of z (c). However, c = 0 is a singular point for that branch of $z_o(c)$ which is represented approximately by (5.14).

We observe - from (8.1) - that if $c \notin \{c_\mu\}$ all the zeros are simple (i.e. of first order). When c approaches one particular c_μ two simple zeros merge to form a double zero (i.e. zero of second order) at $z_{o,\mu}$. For $\mu \neq 0$, Re $z_{o,\mu} > 0$. Therefore, all c_μ $(\mu \neq 0)$ belong to the c-region which gives more than one zero in the right half z-plane.

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